Robust Optimization under Multiband Uncertainty
Supporting practitioners in embedding uncertainty in real-world optimization problems

Motivations
The Bertsimas-Sim Robust Optimization model still constitutes the main reference for optimization under uncertainty in the majority of industrial applications. However, in a series of industrial cooperations and discussions with companies like

our Partners expressed the need for a new Robust Optimization model able to:
- represent arbitrary-shaped distribution of coefficient uncertainty (commonly arising when using historical data about deviations)
- reduce conservatism while preserving the elegant simplicity and accessibility of the Bertsimas-Sim model

Classical Optimization
\[
\begin{align*}
\text{NOMINAL PROBLEM} \\
\text{max} & \quad \sum_{j \in J} c_j x_j \\
\text{s.t.} & \quad \sum_{j \in J} a_{ij} x_j \leq b_i, \quad i \in I \\
& \quad x_j \geq 0, \quad j \in J \\
& \quad x_j \in \mathbb{Z}^+ \\
\end{align*}
\]

ASSUMPTION: the value of each coefficient is known exactly.
However, most real-world problems involve uncertain data.

What would happen if we neglected data uncertainty?
- Optimal solutions may turn out to be of bad quality
- Feasible solutions may become infeasible

Robust Optimization
\[
\begin{align*}
\text{ROBUST COUNTERPART (general form)} \\
\text{max} & \quad \sum_{j \in J} c_j x_j + \text{DEV}(D,x) \\
\text{s.t.} & \quad \sum_{j \in J} a_{ij} x_j + \text{DEV}(D,x) \leq b_i, \quad i \in I \\
& \quad x_j \geq 0, \quad j \in J \\
& \quad x_j \in \mathbb{Z}^+ \\
\end{align*}
\]

KEY CONCEPTS:
- consider only robust solution (i.e., protected against coefficient deviations)
- feasible deviations specified by an uncertainty set \( D \)
- protection imposed by hard robustness constraint
- each robustness constraint includes the max robustness \( \text{DEV}(D,x) \) allowed by D

Single-band Uncertainty
In our industrial cooperations we observed that practitioners tend to analyze data uncertainty by building histograms of observed deviations

EXAMPLE OF HISTOGRAM OF OBSERVED DEVIATIONS FOR A CONSTRAINT

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\begin{align*}
\text{SINGLE-BAND UNCERTAINTY SET (Bertsimas & Sim 2004)} \\
& \quad \text{Hypothesis: each coefficient is symmetrically distributed} \\
& \quad \text{Single symmetric deviation band for each coefficient:} \\
& \quad \left[ a_{ij} - \delta_{a_{ij}}, a_{ij} + \delta_{a_{ij}} \right] \\
& \quad \text{Upper bound} \left( \delta_{a_{ij}} \right) \leq \left| I \right| \text{on the number of coefficients deviating in each constraint}
\end{align*}
\]

\[
\begin{align*}
\text{STRONGPOINT: linear and compact Robust Counterpart} \\
\text{WEAKNESSES:} \\
& \quad \text{symmetric distribution over symmetric deviation range} \\
& \quad \text{focus on extreme deviations} \\
& \quad \text{inner-band behaviour of uncertainty completely neglected}
\end{align*}
\]


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Robust Optimization
\[
\begin{align*}
\text{MAXIMIZATION OF MULTIBAND DEVIATION FOR CONSTRAINT} \\
\text{(BASIC BINARY FORMULATION)} \\
\text{DEV}(D,x) = \max \sum_{j \in J} d_{ij} x_j b_j^k \\
\text{s.t.} \\
& \quad \delta_{a_{ij}} \leq x_j \leq \delta_{a_{ij}}^+ \\
& \quad \delta_{b_i} \leq b_i \leq \delta_{b_i}^+ \\
& \quad \sum_{k \in K} b_j^k \leq 1 \\
& \quad j, k \in J, K
\end{align*}
\]

STRONGPOINTS:
- strongly data-driven uncertainty set
- possibility of modeling non-symmetric deviation distributions

MAIN RESULTS:
1. linear and compact Robust Counterpart
2. separation of robustness cuts = solving a min-cost flow problem
3. solution of Cost-uncertain Binary Program = solving a polynomial number of nominal problems with modified cost coefficients (for a constant number K of bands)

Computational results - highlights
Experiments with real/realistic data of real-world problems show that:
- multiband uncertainty sensibly reduces the price of robustness.
- while granting a comparable level of protection.

Just a single additional band may sensibly reduce solution conservatism

Essential references
3. C. Büsing, F. D’Andreaiovanni, New results about multiband uncertainty in Robust Optimization, SEA 2012

Main original contributions
BASIC IDEA: increase the capacity of modeling uncertainty by adopting multiple deviation bands, extending the single-band uncertainty model by Bertsimas and Sim.

THEORETICAL RESULTS:
- first general theoretical study about multiband uncertainty
- efficient separation of robustness cuts
- compact and linear Robust Counterpart of a MILP
- for cost-uncertain Binary Programs, tractability and approximability are maintained
- data-driven probability bound of constraint violation

PRACTICAL ACHIEVEMENT: reduced price of robustness thanks to the refined representation of the uncertainty

Contacts
Christina Büsing: (email: buising@vri.uni-aachen.de)
Chair of Operations Research, RWTH Aachen University, Kackenstr. 7, 52072 Aachen, Germany

Fabio D’Andreaiovanni (email: d.andreaiovanni@zib.de)
Dept. of Optimization, Zuse-Institut Berlin (ZIB), Takustr. 7, 14195 Berlin, Germany